On 1-Hamilton-connectedness

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A graph G is k-Hamilton-connected (k-hamiltonian) if $G - X$ is Hamilton-connected (hamiltonian) for every set $X \subset V(G)$ with $|X| = k$. We show that

(i) every 5-connected claw-free graph of minimum degree at least 6 is 1-Hamilton-connected,

(ii) as a byproduct, we also show that every 5-connected line graph of minimum degree at least 6 is 3-hamiltonian.

1 Main result

Theorem 1. *Every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.*

In the proof of Theorem 1, we will need the following known facts.

Theorem A [4]. *A line graph G of order at least 3 is Hamilton-connected if and only if $H = L^{-1}(G)$ has an (e_1, e_2)-IDT for any pair of edges $e_1, e_2 \in E(H)$.*

Theorem B [3]. *Let H be a 4-edge-connected graph and let $G = L(H)$. Then G is 2-Hamilton-connected if and only if G is 5-connected.*

Theorem C [5], [6]. *A graph G has k edge-disjoint spanning trees if and only if*

$$|E_0| \geq k(\omega(G - E_0) - 1)$$

for each subset E_0 of the edge set $E(G)$.

In our proof, we will use the hypergraph technique developed in [2], in which vertices of degree 3 are replaced with 3-hyperedges. For the sake of completeness, we repeat here some essential parts from [2]. We include here only basic definitions and facts that are needed for our proof; for more details we refer the reader to the original paper [2].

A *hypergraph* is a pair $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a set of vertices and $E(\mathcal{H})$ is a (multi)set of subsets of $V(\mathcal{H})$ that are called the *hyperedges* of $\mathcal{H}$. A hyperedge of cardinality k is called a *k-hyperedge*. We consider only 3-hypergraphs, i.e., hypergraphs in which each hyperedge is a 2-hyperedge or a 3-hyperedge. Multiple copies of the same hyperedge are allowed. Throughout the rest of this chapter, the symbol $\mathcal{H}$ will always stand for a 3-hypergraph.

In our argument, 3-hypergraphs are obtained from graphs by replacing vertices of degree 3 by hyperedges consisting of their neighbors. Conversely, to every 3-hypergraph $\mathcal{H}$ we assign a graph $Gr(\mathcal{H})$ obtained such that for each 3-hyperedge e of $\mathcal{H}$ we add a vertex v_e and replace e by three edges joining v_e to each vertex of e.

A hypergraph $\mathcal{H}$ is *connected* if for every nonempty proper subset $X \subset V(\mathcal{H})$, there is a hyperedge of $\mathcal{H}$ intersecting both X and $V(\mathcal{H}) \setminus X$. If $\mathcal{H}$ is connected, then an *edge-cut* in $\mathcal{H}$ is an inclusionwise minimal set of hyperedges F such that $\mathcal{H} - F$ is disconnected. For an integer k, $\mathcal{H}$ is k-edge-connected if it is connected and contains no edge-cuts of cardinality less than k. The *degree* of a vertex v is the number of hyperedges incident with v.

For $X \subset V$, we define $\mathcal{H}[X]$ (the *induced subhypergraph of* $\mathcal{H}$ *on* X) as the hypergraph with vertex set X and hyperedge set $E(\mathcal{H}[X]) = \{e \cap X : e \in E(\mathcal{H}) \text{ and } |e \cap X| \geq 2\}$. If $e \cap X = f \cap X$ for distinct hyperedges e, f, we include this hyperedge in multiple copies. Furthermore, we assume a canonical assignment of hyperedges of $\mathcal{H}$ to hyperedges of $\mathcal{H}[X]$. To stress this fact, we always write the hyperedges of $\mathcal{H}[X]$ as $e \cap X$, where $e \in E(\mathcal{H})$.

A *quasigraph* in $\mathcal{H}$ is a pair $(\mathcal{H}, \pi)$, where π is a function assigning to each hyperedge e of $\mathcal{H}$ a set $\pi(e) \subset e$ which is either empty or has cardinality 2. The value $\pi(e)$ is called the representation of e under π. When the underlying hypergraph is clear from the context, we simply speak about a quasigraph π. Quasigraphs will be denoted by lowercase Greek letters. Considering all the nonempty sets $\pi(e)$ as graph edges, we obtain a graph π^* on $V(\mathcal{H})$. We say that hyperedges e with $\pi(e) \neq \emptyset$ are *used* by π; the set of all such hyperedges of $\mathcal{H}$ is denoted by $E(\pi)$, and the edges of the graph π^* are denoted by $E(\pi^*)$.

A quasigraph π is *acyclic* if π^* is a forest and π is a *quasitree* if π^* is a tree. If e is a hyperedge of $\mathcal{H}$, then $\pi - e$ is the quasigraph obtained from π by changing the value at e to $\emptyset$. The *complement* $\overline{\pi}$ of π is the spanning subhypergraph of $\mathcal{H}$ consisting of all the hyperedges of $\mathcal{H}$ not used by π. Note that $\overline{\pi}$ is not a quasigraph, and since π includes the information about its underlying hypergraph $\mathcal{H}$, we can speak about $\overline{\pi}$ without specifying $\mathcal{H}$.

For $X \subset V(\mathcal{H})$, the π*-section of* $\mathcal{H}$ *at* X is the hypergraph $\mathcal{H}[X]^\pi$ with $V(\mathcal{H}[X]^\pi) = X$ and $E(\mathcal{H}[X]^\pi) = \{e \cap X : e \in E(\mathcal{H}) \text{ is such that } |e \cap X| \geq 2 \text{ and } \pi(e) \subset X\}$. The quasigraph π in $\mathcal{H}$ naturally determines a quasigraph $\pi[X]$ in $\mathcal{H}[X]^\pi$, defined by $(\pi[X])(e \cap X) = \pi(e)$, where $e \in E(\mathcal{H})$ and $e \cap X$ is any hyperedge of $\mathcal{H}[X]^\pi$. The quasigraph $\pi[X]$ is called the *quasigraph induced by* π *on* X. Note that whenever we speak about the complement of $\pi[X]$, it is, in accordance with the definition, its complement in $\mathcal{H}[X]^\pi$.

A quasigraph π has *tight complement* (in $\mathcal{H}$) if π satisfies one of the following:

(a) $\overline{\pi}$ is connected, or

(b) there is a partition $V(\mathcal{H}) = X_1 \cup X_2$ such that for $i = 1, 2$, X_i is nonempty and $\pi[X_i]$ has tight complement (in $\mathcal{H}[X_i]^\pi$); furthermore, there is a hyperedge $e \in E(\pi)$ such that $\pi(e) \subset X_1$ and $e \cap X_2 \neq \emptyset$.

A set $X \subset V(\mathcal{H})$ is π*-solid* (in $\mathcal{H}$), if $\pi[X]$ is a quasitree with tight complement in $\mathcal{H}[X]^\pi$.

Let $\mathcal{P}$ be a partition of $V(\mathcal{H})$. An edge $e \in E(\mathcal{H})$ is $\mathcal{P}$*-crossing* if e intersects at least two classes of $\mathcal{P}$ and, for a $\mathcal{P}$-crossing edge e, $e/\mathcal{P}$ is the set of all classes $P \in \mathcal{P}$ with $e \cap P \neq \emptyset$.

The *contraction of* $\mathcal{P}$ is the operation resulting in the hypergraph $\mathcal{H}/\mathcal{P}$ with $V(\mathcal{H}/\mathcal{P}) = \mathcal{P}$ and $E(\mathcal{H}/\mathcal{P}) = \{e/\mathcal{P} : e \text{ is } \mathcal{P}\text{-crossing}\}$. Thus, $\mathcal{H}/\mathcal{P}$ is a 3-hypergraph, possibly with multiple hyperedges.

If π is a quasigraph in $\mathcal{H}$, we define $\pi/\mathcal{P}$ as the quasigraph in $\mathcal{H}/\mathcal{P}$ consisting of the hyperedges $e/\mathcal{P}$ such that $\pi(e)$ is $\mathcal{P}$-crossing; the representation is defined by $(\pi/\mathcal{P})(e/\mathcal{P}) = \pi(e)/\mathcal{P}$. Obviously, the complement of $\pi/\mathcal{P}$ in $\mathcal{H}/\mathcal{P}$ is denoted by $\overline{\pi/\mathcal{P}}$.

Finally, if π is a quasigraph in $\mathcal{H}$, then a partition $\mathcal{P}$ of $V(\mathcal{H})$ is said to be π-*skeletal* if every $X \in \mathcal{P}$ is π-solid and the complement of $\pi/\mathcal{P}$ in $\mathcal{H}/\mathcal{P}$ is acyclic.

The following lemma is a special case of the Skeletal lemma (Lemma 17 of [2]).

Lemma D [2]. *Every 3-hypergraph $\mathcal{H}$ contains an acyclic quasigraph σ such that there is a σ-skeletal partition $\mathcal{S}$ of $V(\mathcal{H})$.*

Proof of Theorem 1. If G is a counterexample to Theorem 1 and $\overline{G}$ is a 1HC-closure of G, then $\overline{G}$ is also a counterexample to Theorem 1; hence it is sufficient to prove Theorem 1 for line graphs (of multigraphs). Thus, let $G = L(H)$. If H is 4-edge-connected, then the statement follows from Theorem B, hence it remains to prove the theorem in the case when $G = L(H)$ and H is not 4-edge-connected. By Theorem A, we need to show that, for any $e_1, e_2, e_3 \in E(H)$, the graph $H - e_3$ has an (e_1, e_2)-IDT.

By the minimum degree assumption, every edge of H is of weight at least 6, and by the connectivity assumption, H is essentially 5-edge-connected. By the assumption that H is not 4-edge-connected, H must contain vertices of degree 3, and since H is essentially 5-edge-connected, $V_{\leq 3}(H)$ is an independent set in H. Thus, it is sufficient to find in $H - e_3$ an (e_1, e_2)-trail spanning all vertices in $V_{\geq 4}(H)$. For finding such a trail, we use the concept of an X-join: for $X \subset V(H)$, an X-*join in* H is a subgraph H_0 of H such that a vertex of H is in X if and only if its degree in H_0 is odd (in particular, $\emptyset$-joins are eulerian subgraphs.)

For each edge e of H, fix a vertex u_e of degree at least 4 in H (which exists since $V_{\leq 3}(H)$ is independent), and for $e_1, e_2 \in E(H)$ set

$$X(e_1, e_2) = \begin{cases} \{u_{e_1}, u_{e_2}\} & \text{if } u_{e_1} \neq u_{e_2}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now, if the graph $H - e_1 - e_2 - e_3$ contains a connected $X(e_1, e_2)$-join J spanning all of $V_{\geq 4}(H)$, then, by the classical observation of Euler, all the edges of J can be arranged in a trail T_J with first edge incident with u_{e_1} and last edge incident with u_{e_2}. Adding e_1 and e_2, we obtain a desired (e_1, e_2)-trail T in $H - e_3$ spanning $V_{\geq 4}(H)$ (if $u_1 = u_2$, we use the fact that u_1 is incident with an edge of T_J). Thus, our task is reduced to finding a connected $X(e_1, e_2)$-join in $H - e_1 - e_2 - e_3$ spanning all vertices in $V_{\geq 4}(H)$. For finding such a join we use the hypergraph technique.

For this, we need the following observation. Suppose that there is a vertex $u \in V_{\leq 3}(H)$ having at most 2 neighbors, and let H' be the graph obtained by removing u if u has 1 neighbor, or by suppressing u (i.e., removing u and adding an edge joining its neighbors) if u has 2 neighbors, respectively. By the connectivity and minimum degree assumptions, $V_{\geq 4}(H') = V_{\geq 4}(H)$ and if, for any $f_1, f_2, f_3 \in E(H')$, $H' - f_3$ contains an (f_1, f_2)-trail spanning $V_{\geq 4}(H')$, we can easily find a desired (e_1, e_2)-trail in $H - e_3$ spanning $V_{\geq 4}(H)$. Thus, we can suppose that $V_1(H) = V_2(H) = \emptyset$ and every vertex in $V_3(H)$ has 3 neighbors.

We can now define the 3-hypergraph $\mathcal{H}$ with vertex set $V(\mathcal{H}) = V_{\geq 4}(H)$; the hyperedges of $\mathcal{H}$ are of two types:

- the edges of H with both endvertices in $V(\mathcal{H})$,
- 3-hyperedges consisting of the neighbors of any vertex in $V_3(H)$.

Recall that every vertex in $V_3(H)$ has three distinct neighbors and these are in $V(\mathcal{H}) = V_{\geq 4}(H)$ since $V_3(H)$ is independent. Also note that clearly $H = Gr(\mathcal{H})$.

We show that $\mathcal{H}$ has properties that will be of importance for us (Claims 1 and 2 are proved in [2]; for the sake of completeness, we include their proofs here as well).

<u>Claim 1.</u> *The hypergraph $\mathcal{H}$ is 4-edge-connected.*

<u>Proof.</u> Suppose that this is not the case, let F be an inclusionwise minimal edge-cut in $\mathcal{H}$ with $|F| \leq 3$ and let A be the vertex set of a component of $\mathcal{H} - F$. Let $e \in F$. By the minimality of F, $|e - A| \geq 1$. We assign to e an edge e' of H, defined as follows:

- if $|e| = 2$, then $e' = e$,
- if $|e| = 3$ and $e \cap A = \{u\}$, then $e' = uv_e$,
- if $|e| = 3, |e \cap A| = 2$ and $e - A = \{u\}$, then $e' = uv_e$.

Then $F' := \{e' : e \in F\}$ is an edge-cut in H and since H is essentially 5-edge-connected, F' is a trivial edge-cut. Since $|F'| \leq 3$, A contains a vertex of degree 3 (in H), a contradiction. □

<u>Claim 2.</u> *No 3-hyperedge of $\mathcal{H}$ is included in an edge-cut of size 4 in $\mathcal{H}$.*

<u>Proof.</u> Let F be an edge-cut of size 4 in $\mathcal{H}$. As in the proof of Claim 1, we consider the corresponding edge-cut F' in H. Since H is essentially 5-edge-connected, one component of $H - F'$ consists of a single vertex w whose degree in H is 4. Assuming that F includes a 3-hyperedge e, we find that in H, w has a neighbor v of degree 3. Thus, the weight of the edge vw is 5, a contradiction. □

Let $e_1, e_2, e_3 \in E(H)$ be the given edges, and let w_i, $i = 1, 2, 3$, be the vertex of e_i distinct from u_{e_i}. We define a 3-hypergraph $\mathcal{H}^{\{e_1,e_2,e_3\}}$ by the following construction.

(1) If some two (possibly all three) of e_1, e_2, e_3 have a common vertex of degree 3, i.e., $w_i = w_j$ for some $i, j \in \{1, 2, 3\}$, then let $\mathcal{H}_1$ be the hypergraph obtained from $\mathcal{H}$ by removing the 3-hyperedge corresponding to the vertex $w_i = w_j$;
otherwise set $\mathcal{H}_1 = \mathcal{H}$
(note that, after this step, $|\{w_i : w_i \in Gr(\mathcal{H}_1)\}| \in \{0, 1, 3\}$).

(2) Let $\mathcal{H}^{\{e_1,e_2,e_3\}}$ be the hypergraph obtained from $\mathcal{H}_1$ by performing the following for every $w_i \in Gr(\mathcal{H}_1)$:

(2a) if w_i has degree 3 in $\mathcal{H}$, then the 3-hyperedge e_{w_i} of $\mathcal{H}_1$ corresponding to w_i is replaced by the 2-hyperedge $e_{w_i} - \{u_{e_i}\}$,

(2b) otherwise, the 2-hyperedge e_i of $\mathcal{H}_1$ is deleted (note that, unlike $\mathcal{H}$, the hypergraph $\mathcal{H}^{\{e_1,e_2,e_3\}}$ can contain vertices of degree 3).

Then $Gr(\mathcal{H}^{\{e_1,e_2,e_3\}})$ is either $H - e_1 - e_2 - e_3$, or is obtained from $H - e_1 - e_2 - e_3$ by removing a pendant edge (in the case when exactly two of e_1, e_2, e_3 share a vertex of degree 3). In both cases, $V_{\geq 4}(Gr(\mathcal{H}^{\{e_1,e_2,e_3\}})) = V_{\geq 4}(H - e_1 - e_2 - e_3)$, and $X(e_1, e_2)$-joins spanning this set are the same in $Gr(\mathcal{H}^{\{e_1,e_2,e_3\}})$ and in $H - e_1 - e_2 - e_3$.

By the previous observations, our task is reduced to finding a connected $X(e_1, e_2)$-join in $Gr(\mathcal{H}^{\{e_1,e_2,e_3\}})$ spanning all vertices in $V(\mathcal{H})$. If $\mathcal{H}^{\{e_1,e_2,e_3\}}$ has a quasitree with tight complement, then the existence of such a join is guaranteed by the following result (Lemma 28 of [2]).

Lemma E [2]. *Let $\mathcal{H}$ be a 3-hypergraph containing a quasitree π with tight complement, and let $X \subset V(\mathcal{H})$. Then there is a quasigraph τ such that $E(\pi)$ and $E(\tau)$ are disjoint, and $\pi^* + \tau^*$ is a connected X-join in $Gr(\mathcal{H})$ spanning all vertices in $V(\mathcal{H})$.*

It should be noted here that, by Claims 1 and 2, $\mathcal{H}$ satisfies the assumptions of the following result (Theorem 5 of [2]), which therefore guarantees the existence of a quasitree with tight complement in $\mathcal{H}$.

Theorem F [2]. *Let $\mathcal{H}$ be a 4-edge-connected 3-hypergraph. If no 3-hyperedge in $\mathcal{H}$ is included in any edge-cut of size 4, then $\mathcal{H}$ contains a quasitree with tight complement.*

However, in $\mathcal{H}^{\{e_1,e_2,e_3\}}$, a quasitree with tight complement does not have to exist due to the fact that $\mathcal{H}^{\{e_1,e_2,e_3\}}$ was obtained from $\mathcal{H}$ by removing some hyperedges, hence reducing degrees of some vertices. In such case, the following result gives the existence of a quasigraph and a skeletal partition $\mathcal{S}$ of $V(\mathcal{H})$ with a special structure, and we can use $\mathcal{S}$ for constructing the desired join.

Theorem 2. *Let $\mathcal{H}$ be a 4-edge-connected 3-hypergraph with at least one 3-hyperedge such that no 3-hyperedge of $\mathcal{H}$ is included in an edge-cut of size 4 and $Gr(\mathcal{H})$ is essentially 5-edge-connected. Let $e_1, e_2, e_3 \in E(Gr\mathcal{H})$ and set $\mathcal{H}' = \mathcal{H}^{\{e_1,e_2,e_3\}}$. If $\mathcal{H}'$ has no quasitree with tight complement, then there is a quasigraph π in $\mathcal{H}'$ and a π-skeletal partition $\mathcal{S}$ of $V(\mathcal{H}')$ such that:*

(i) one of the classes of $\mathcal{S}$ is a trivial class $\{x\}$,
(ii) the degree of x in $\mathcal{H}$ is 4,
(iii) e_1, e_2, e_3 are 2-hyperedges in $\mathcal{H}$ and each of e_1, e_2, e_3 is incident (in $\mathcal{H}$) with x.

Proof. By Lemma D, $\mathcal{H}'$ contains an acyclic quasigraph π and a π-skeletal partition $\mathcal{P}$. By the assumption, $\mathcal{H}'$ has no quasitree with tight complement, hence $\mathcal{P}$ is nontrivial.

Assume that $\mathcal{H}/\mathcal{P}$ has n vertices (i.e., $|\mathcal{P}| = n$) and m hyperedges. Let m' denote the number of hyperedges in $\mathcal{H}'/\mathcal{P}$, m'_k the number of k-hyperedges of $\pi/\mathcal{P}$ and $\overline{m'_k}$ the number of k-hyperedges of $\overline{\pi/\mathcal{P}}$, $k \in \{2, 3\}$. Thus, $m' = m'_2 + m'_3 + \overline{m'_2} + \overline{m'_3}$.

Since $\overline{\pi/\mathcal{P}}$ is acyclic, the graph $Gr(\overline{\pi/\mathcal{P}})$ is a forest. As $Gr(\overline{\pi/\mathcal{P}})$ has $n + \overline{m'_3}$ vertices and $\overline{m'_2} + 3\overline{m'_3}$ edges, we find that

$$\overline{m'_2} + 2\overline{m'_3} \leq n - 1. \tag{1}$$

Since $\mathcal{P}$ is π-solid and π is an acyclic quasigraph, we know that $m_2' + m_3' \leq n - 1$. Moreover, by the assumption that π is not a quasitree with a tight complement, either this inequality or (1) is strict. Summing the two, we obtain

$$m' + \overline{m_3'} \leq 2n - 3. \tag{2}$$

For an arbitrary hypergraph $\mathcal{H}^*$, let $s(\mathcal{H}^*)$ denote the sum of degrees of all its vertices. By the construction of $\mathcal{H}'$ from $\mathcal{H}$, the operations (1), (2a) and (2b) can decrease the degree sum by at most 6 (if all the edges e_1, e_2, e_2 are hyperedges of size 2; otherwise the decrease is less than 6). Hence we have

$$s(\mathcal{H}'/\mathcal{P}) \geq s(\mathcal{H}/\mathcal{P}) - 6. \tag{3}$$

Set $n_4 = |V_4(\mathcal{H}/\mathcal{P})|$ and $n_{5^+} = |V_{\geq 5}(\mathcal{H}/\mathcal{P})|$. Since $\mathcal{H}$ is 4-edge-connected, we have $n_4 + n_{5^+} = n$ and

$$s(\mathcal{H}/\mathcal{P}) \geq 4n_4 + 5n_{5^+}. \tag{4}$$

By simple counting,

$$s(\mathcal{H}'/\mathcal{P}) = 2(m_2' + \overline{m_2'}) + 3(m_3' + \overline{m_3'}) = 2m' + m_3' + \overline{m_3'}. \tag{5}$$

Combining (4), (3) and (5), we have

$$4n_4 + 5n_{5^+} - 6 \leq s(\mathcal{H}/\mathcal{P}) - 6 \leq s(\mathcal{H}'/\mathcal{P}) = 2m' + m_3' + \overline{m_3'},$$

from which

$$4n_4 + 5n_{5^+} - 6 \leq 2m' + m_3' + \overline{m_3'}. \tag{6}$$

From (2) we have

$$2m' + 2\overline{m_3'} \leq 4(n_4 + n_{5^+}) - 6,$$

which, using (6), gives

$$2m' + 2\overline{m_3'} \leq 2m' + m_3' + \overline{m_3'} - n_{5^+},$$

or, equivalently,

$$\overline{m_3'} + n_{5^+} \leq m_3'. \tag{7}$$

Suppose that $m_3' > 0$. Let $T' = (\pi/\mathcal{P})^*$ be the forest on $\mathcal{P}$ which represents $\pi/\mathcal{P}$. In each component of T', containing an edge corresponding to a 3-hyperedge of $(\pi/\mathcal{P})$, choose a root in that edge and direct the edges of T' away from it. To each 3-hyperedge $e \in E(\pi/\mathcal{P})$, assign the head $h(e)$ of the arc $\pi(e)$. By the assuptions of the theorem, no edge-cut of size 4 contains a 3-hyperedge, so $h(e)$ is a vertex of degree at least 5 and, by the same argument, the root is also of degree at least 5. At the same time, since each vertex is the head of at most one arc in the directed forest, it gets assigned to at most one hyperedge. From this we have

$$n_{5^+} \geq m_3' + 1. \tag{8}$$

Combining (7) and (8), we obtain $\overline{m_3'} + n_{5^+} \leq m_3' \leq n_{5^+} - 1$, implying $\overline{m_3'} + 1 \leq 0$, a contradiction.

Hence we have $m'_3 = 0$, and from (7) we have $n_{5^+} = 0$. Since every vertex of a 3-hyperedge is of degree at least 5, we have also $m_3 = 0$. Thus, $\mathcal{H}/\mathcal{P}$ is 4-regular and all its hyperedges are of size 2.

By the assumption, $\mathcal{H}$ has at least one 3-hyperedge, hence at least one element of $\mathcal{P}$ is nontrivial. If there are two nontrivial elements of $\mathcal{P}$, say, P_1 and P_2, then the edges connecting P_1 to the rest of $\mathcal{H}$ form an essential edge-cut of size 4 in $Gr(\mathcal{H})$, contradicting the assumption that $Gr(\mathcal{H})$ is essentially 5-edge-connected. Hence exactly one element of $\mathcal{P}$, say, P_1, is nontrivial. Since $\mathcal{H}/\mathcal{P}$ is 4-regular and $\mathcal{P}$ has at least 2 elements, $|\mathcal{P}| = 2$ and the second element of $\mathcal{P}$, P_2, is trivial. Set $P_2 = \{x\}$.

If x is connected in $\mathcal{H}'$ to $\mathcal{H}'[P_1]$ with at least two edges, we easily extend $\pi[P_1]$ to a quasitree with tight complement in $\mathcal{H}'$, a contradiction. Hence x is connected to $\mathcal{H}'[P_1]$ in $\mathcal{H}'$ with exactly one edge. Since x is of degree 4 in $\mathcal{H}$, x is incident with each of e_1, e_2, e_3. ■

Now we can complete the proof of Theorem 1. Let u_{e_i} be the vertex of e_i different from x, $i = 1, 2$. By the structure of $\mathcal{H}$ described in Theorem 2, an $X(e_1, e_2)$-join in $Gr(\mathcal{H}'[P_1])$ (which exists by Lemma E and since P_1 is π-solid) has the required properties. ■

2 Concluding remarks

By a slight modification of the proof of Theorem 1, we can also obtain the following result.

Theorem 3. *Every 5-connected line graph graph with minimum degree at least 6 is 3-hamiltonian.*

Proof of Theorem 3 is similar to the proof of Theorem 1 with the only difference that instead of proving the existence of a $X(e_1, e_2)$-join, we find in $Gr(\mathcal{H}'[P_1])$ a $\emptyset$-join. If H is not 4-edge-connected, then the existence of a $\emptyset$-join in $Gr(\mathcal{H}'[P_1])$ follows by Theorem 2 in the same way as in Section 1; the case when H is 4-edge-connected has to be treated in a slightly different way. For this, we will need to recall some more concepts and facts.

A graph G is *collapsible* if, for any even subset $R \subset V(G)$, G has a spanning connected subgraph F such that $O(F) = R$, where $O(F)$ denotes the set of vertices of odd degree in F. The *reduction* of G is the graph obtained from G by contracting every maximal collapsible subgraph of G to a distinct vertex. Clearly, if a graph G is collapsible, then G has an X-join for any $X \subset V(G)$. For a graph G, let $f(G)$ denote the minimum number of edges that have to be added to G so that the resulting graph has two edge-disjoint spanning trees. We will need the following fact by Catlin et al. [1].

Theorem G [1]. *Let G be a connected graph with $f(G) \leq 2$. Then G is collapsible or the reduction of G is either K_2 or a $K_{2,t}$ for some $t \geq 1$.*

Thus, suppose now that H is 4-edge-connected, and set $H' - H - e_1$ and $H'' = H' - e_2 - e_3 = H - e_1 - e_2 - e_3$. If H' has two edge-disjoint spanning trees, then, by Theorem G, either H'' is collapsible and we are done, or the reduction of H'' is either K_2

or a $K_{2,t}$ for some $t \geq 1$. However, the second case is impossible since adding three edges to a $K_{2,t}$ can never create a reduction of a 4-edge-connected graph, and if the reduction of H'' is K_2, we find a desired $\emptyset$-join in the same way as in the proof of Theorem 1.

Thus, it remains to show that $H' = H - e_1$ has 2 edge-disjoint spanning trees. By Theorem C, we need to show that $|E_0| \geq 2(\omega(H' - E_0) - 1)$ for any $E_0 \subset E(H')$. Since H is 4-edge-connected, every component of $H - E_0$ is connected to the rest of $H - E_0$ by at least 4 edges, implying $2|E_0| \geq 4\omega(H - E_0)$, from which $|E_0| \geq 2\omega(H - E_0)$ (for any $E_0 \subset E(H)$, hence also for any $E_0 \subset E(H')$). Since $H' = H - e_1$, we have $\omega(H - E_0) \geq \omega(H' - E_0) - 1$, implying $|E_0| \geq 2\omega(H - E_0) \geq 2(\omega(H' - E_0) - 1)$, as required. ■

We are not able to extend Theorem 3 to claw-free graphs since a closure concept that would make this possible is not known so far.

References

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